VARIATIONAL FORMULATION OF THE HYDRODYNAMICS OF A CONCENTRATED GAS—SOLID SYSTEM AT HIGH ARCHIMEDES NUMBERS

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The local Glandsdorff—Prigogine potential is formulated for an isothermal fluidized bed. It is shown that this variational formulation makes it possible to describe the hydrodynamics of a fluidized bed on the basis of a numerical solution of the problem of minimizing the resulting local potential for the case of a two-dimensional bed.

Two-phase concentrated ($Ar \geq 10^2$) systems of fluidized beds of the gas—solid type are widely used in various industrial processes. The gas which filters through the bed of particles moves in a nonuniform manner; gas bubbles devoid of particles break through the bed. The result is a sharp degradation of the interfacial exchange in the system, and the efficiency of the corresponding equipment is reduced. This circumstance is responsible for the interest in the phase motion in such systems.

A rigorous description of a two-phase homogeneous system can obviously be obtained by statistical methods, but the models which have been constructed at this point are not useful for obtaining practical information. Phenomenological approaches are thus of interest. In such an approach the system is treated as consisting of two mutually penetrating continua, for which mass, momentum, and energy conservation laws are written [1, 2]. The system of conservation equations is nonlinear and, as was shown in [2], unstable against small perturbations. This circumstance poses insurmountable difficulties for a numerical solution; some additional hypothesis is required.

One such hypothesis is the concept of a local potential, which has been used successfully to solve several problems of practical importance [3, 4]. This approach reduces to seeking a minimum of a functional reflecting the stability of the fluctuations in the dynamic variables which occur in the system.

Below we attempt to apply this approach to the hydrodynamics of a concentrated two-phase system (a fluidized bed) in which the phase concentrations vary significantly in space and time.

We consider an isothermal fluidized bed consisting of solid particles of uniform diameter, for which we have $Ar > 10^2$. A gas is filtering upward through the bed. The particle concentration in the system lies in the range $0.3 < \varepsilon < 0.7$. We denote the velocities of the gas and the solid particles by v_i and w_i, respectively. Each particle experiences the gravitational force and a friction force with the gas. There is no difficulty in determining the first of these forces; the friction force can be found from Newton's friction law

$$F_i = \mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) f n_k, \tag{1}$$

where n_k are the direction cosines of the normal to some area with an arbitrary orientation in the bed. The friction surface area (the common part of the space of the gas and solid particles) is proportional to $\varepsilon(1-\varepsilon)$; i.e., we have $f \sim \varepsilon(1-\varepsilon)$. In the linear approximation, using $\partial v_i/\partial x_k \to 0$ for $\varepsilon \to 0$, in the limit, we have $n_k(\partial v_i/\partial x_k) \sim \varepsilon(v_i - w_i)$, so that the friction forces becomes

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$$F_i = k\varepsilon^2 (1-\varepsilon)(v_i - w_i), \tag{2}$$

where the coefficient k is determined from the condition that the solid particles are suspended by the gas at the beginning of the fluidization.

In the system under consideration here we have $(\rho_f/\rho_s) \sim 10^{-3}$, so we can neglect changes in the kinetic and potential energy of the gas in the gravitational field in comparision with the analogous changes for the particles.

We further assume that in the unsteady state the gas and the solid particles have different pressures, p_f and p_s, respectively, which become equal in the stable (equilibrium) state ($p_f^0 = p_s^0 = p^0$).

Let us formulate a local potential for this system:

$$\begin{aligned} L = \int_V \int_t \Big\{ & -v_i^0 (1-\varepsilon^0) \frac{\partial p_f}{\partial x_i} - p_f \frac{\partial \varepsilon^0}{\partial t} - w_i^0 \varepsilon^0 \frac{\partial p_s}{\partial x_i} + p_s \frac{\partial \varepsilon^0}{\partial t} - \\ & - \rho_s \varepsilon^0 w_i^0 w_j^0 \frac{\partial w_i}{\partial x_j} + \frac{1}{2} \rho_s \varepsilon^0 w_j^0 \frac{\partial w_i^2}{\partial x_j} + \rho_s g \varepsilon^0 w_z + \\ & + v_i \frac{\partial p^0}{\partial x_i} + \rho_s \varepsilon^0 w_i \frac{\partial w_i^0}{\partial t} + \\ & + k {\varepsilon^0}^2 (1-\varepsilon^0)(v_i^0 - w_i^0)(v_i - w_i) \Big\} dV dt. \end{aligned} \tag{3}$$

The quantities with subscript "0" refer to the steady state, while quantities without subscripts refer to the unsteady state. Here we are assuming that the deviations from the steady (stable) state are very small. In the steady state the integral L is minimal and can be identified with the rate at which entropy is produced near the stable state.

To show that functional (3) actually describes the hydrodynamics of a fluidized bed we evaluate the variation δL of this functional, assuming that quantities with subscript "0" are not varied [3]:

$$\begin{aligned} \delta L = \int_V \int_t \Big\{ & - v_i^0 (1-\varepsilon^0) \delta \frac{\partial p_f}{\partial x_i} - \frac{\partial \varepsilon^0}{\partial t} \delta p_f - w_i^0 \varepsilon^0 \delta \frac{\partial p_s}{\partial x_i} + \\ & + \frac{\partial \varepsilon^0}{\partial t} \delta p_s - \rho_s \varepsilon^0 w_i^0 w_j^0 \delta \frac{\partial w_i}{\partial x_j} + \frac{1}{2} \rho_s \varepsilon^0 w_j^0 \delta \frac{\partial w_i^2}{\partial x_j} + \\ & + \rho_s g \varepsilon^0 \delta w_z + \frac{\partial p^0}{\partial x_i} \delta v_i + \rho_s \varepsilon^0 \frac{\partial w_i^0}{\partial t} \delta w_i + \\ & + k {\varepsilon^0}^2 (1-\varepsilon^0)(v_i^0 - w_i^0) \delta (v_i - w_i) \Big\} dV dt. \end{aligned} \tag{4}$$

After some straightforward manipulations involving the use of the Gauss theorem and the assumption that these functions are given at the boundaries of region V, we find

$$\begin{aligned} \delta L = \int_V \int_t \Big[& \Big\{ - \frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i} [v_i (1-\varepsilon)] \Big\} \delta p_f + \\ & + \Big\{ \frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i} (w_i \varepsilon) \Big\} \delta p_s + \Big\{ \rho_s \varepsilon \frac{\partial w_i}{\partial t} + \rho_s \varepsilon w_j \frac{\partial w_i}{\partial x_j} + \\ & + \rho_s \varepsilon g - k\varepsilon^2 (1-\varepsilon)(v_i - w_i) \Big\} \delta w_i + \Big\{ \frac{\partial (p \delta_{ij})}{\partial x_j} + \\ & + k\varepsilon^2 (1-\varepsilon)(v_i - w_i) \Big\} \delta v_i \Big] dV dt. \end{aligned} \tag{5}$$

After the variation, we equated the quantities with and without the subscript "0."

Accordingly, we find that the condition

$$\delta L = 0 \tag{6}$$

implies

$$- \frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i} [v_i (1-\varepsilon)] = 0,$$

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}(w_i \varepsilon) = 0,$$

$$\rho_s \varepsilon \frac{\partial w_i}{\partial t} + \rho_s \varepsilon w_j \frac{\partial w_i}{\partial x_j} + \rho_s \varepsilon g - k\varepsilon^2 (1-\varepsilon)(v_i - w_i) = 0, \tag{7}$$

$$\frac{\partial}{\partial x_j}(p\delta_{ij}) + k\varepsilon^2 (1-\varepsilon)(v_i - w_i) = 0.$$

The Lagrange–Euler equations in (7), which describe the extremum of functional (3), are the conservation equations for the masses and momenta of the phases in this two-phase system.

Accordingly, this variational problem for seeking the minimum of functional (3) is a method for describing the hydrodynamics of a two-phase flow which can be used instead of the system of mass and momentum conservation equations for the phases in (7).

The variational formulation is known [5] to have important advantages in a numerical solution, especially if the problem is nonlinear, is multidimensional, and has a large number of unknown functions. Furthermore, this formulation makes it possible to obtain more information about the system than can be obtained with Eqs. (7). For example, for the case of a fluidized bed we can obtain from the variational principle an additional equation for determining the bed height H, which does not remain constant as the gas filters through the bed.

First let us put the problem in a slightly more concrete form. We consider a fluidized bed of particles (the velocity corresponding to the beginning of fluidization is v_0) in a cylindrical column of radius R; the bed height is H. Gas filters through the bed at a velocity Nv_0, where N is the fluidization number. We treat the three-dimensional problem, in which all properties of the bed depend on the coordinates z and r and the time t. We write the functional in (3) in the dimensionless form

$$\begin{aligned} L' = \int_0^1\int_0^1\int_0^1 \Big\{ & -Ev_z^{0'}(1-\varepsilon^0)\frac{\partial p_f'}{\partial z'} - E\gamma v_r^{0'}(1-\varepsilon^0)\frac{\partial p_f'}{\partial r'} - \\ & - ENp_f'\frac{\partial \varepsilon^0}{\partial t'} - Ew_z^{0'}\varepsilon^0\frac{\partial p_s'}{\partial z'} - E\gamma w_r^{0'}\frac{\partial p_s'}{\partial r'} + \\ & + ENp_s'\frac{\partial \varepsilon^0}{\partial t'} - \varepsilon^0\Big[(w_z^{0'})^2\frac{\partial w_z'}{\partial z'} + \\ & + w_z^{0'}w_r^{0'}\frac{\partial w_r'}{\partial z'}\Big] + \frac{\varepsilon^0}{2}\Big[w_z^{0'}\frac{\partial (w_z')^2}{\partial z'} + w_z^{0'}\frac{\partial (w_r')^2}{\partial z'}\Big] + \\ & + \frac{1}{Fr}\varepsilon^0 w_z' - \gamma\varepsilon^0\Big[w_z^{0'}w_r^{0'}\frac{\partial w_z'}{\partial r'} + (w_r^{0'})^2\frac{\partial w_r'}{\partial r'}\Big] + \\ & + \frac{\gamma}{2}\varepsilon^0\Big[w_r^{0'}\frac{\partial (w_z')^2}{\partial r'} + w_r^{0'}\frac{\partial (w_r')^2}{\partial r'}\Big] + N\varepsilon^0\Big(\frac{\partial w_z^{0'}}{\partial t'}w_z' + \\ & + \frac{\partial w_r^{0'}}{\partial t'}w_r'\Big) + Ev_z'\frac{\partial p^{0'}}{\partial z'} + E\gamma v_r'\frac{\partial p^{0'}}{\partial r'} + \\ & + k_1 {\varepsilon^0}^2(1-\varepsilon^0)(v_i^{0'} - w_i^{0'})(v_i' - w_i')\Big\} r'dr'dz'dt'. \end{aligned} \tag{8}$$

Here we have introduced the dimensionless variables

$$\frac{r}{R} = r'; \quad \frac{t}{T_0} = t'; \quad \frac{z}{H} = z'; \quad \frac{v_i}{v_0} = v_i'; \quad \frac{w_i}{v_0} = w_i'; \quad \frac{p}{p_0} = p'.$$

An approximate solution of this problem has been found by the Ritz method [5]. The coordinate functions were chosen to be

$$\varphi_{k,n}(z', r', t') = \sin k\left(2\pi t' - \frac{2\pi z'}{a}\right)\sin n\left(\frac{2\pi r'}{b}\right),$$

$$\psi_{k,n}(z', r', t') = \cos k\left(2\pi t' - \frac{2\pi z'}{a}\right)\sin n\left(\frac{2\pi r'}{b}\right)$$

$$(k, n = 1, \ldots m, \ldots).$$

In the first approximation we have

$$\varepsilon=\begin{cases} a_1^\varepsilon+a_2^\varepsilon\sin\left(2\pi t'-\dfrac{2\pi z'}{a}\right)\sin\dfrac{2\pi r'}{b}, & 0<r'<b,\\ a_3^\varepsilon, & b<r'<1.\end{cases}$$

$$v_z'=\begin{cases} N+a_2^v\sin\left(2\pi t'-\dfrac{2\pi z'}{a}\right)\sin\dfrac{2\pi r'}{b}, & 0<r'<b,\\ a_3^v, & b<r'<1,\end{cases}$$

$$v_r'=\begin{cases} a_4^v\sin 2\left(2\pi t'-\dfrac{2\pi z'}{a}\right)\sin^*\dfrac{2\pi r'}{b}, & 0<r'<b,\\ 0, & b<r'<1,\end{cases}$$

$$w_z'=\begin{cases} a_1^w+a_2^w\sin\left(2\pi t'-\dfrac{2\pi z'}{a}\right)\sin\dfrac{2\pi r'}{b}, & 0<r'<b,\\ a_3^w, & b<r'<1,\end{cases}$$

$$w_r'=\begin{cases} a_4^w\sin 2\left(2\pi t'-\dfrac{2\pi z'}{a}\right)\sin^*\dfrac{2\pi r'}{b}, & 0<r'<b,\\ 0, & b<r'<1,\end{cases}$$

$$p'=\begin{cases} (1-a_1^p)z'+a_1^p+a_2^p\cos\left(2\pi t'-\dfrac{2\pi z'}{a}\right)\sin\dfrac{2\pi r'}{b}, & 0<r'<b,\\ (1-a_1^p)z'+a_1^p, & b<r'<1,\end{cases}$$

where

$$\sin^*(x)=\begin{cases} \sin x, & 0<x<\dfrac{b}{4};\ \dfrac{b}{2}<x<\dfrac{3b}{4},\\ -\sin x, & \dfrac{b}{4}<x<\dfrac{b}{2};\ \dfrac{3b}{4}<x<b.\end{cases}$$

In this approach we assume the following picture for the motion [6]: gas bubbles break through and solid material moves upward at the center of the bed, while at the walls the material is descending and there are no gas bubbles.

After the trial functions are substituted into (8) and the integration is carried out, the functional becomes an algebraic function of a_i^j and a_i^{0j}:

$$L'=L'(a_i^j,\ a_i^{0j},\ H,\ \ldots); \tag{9}$$

its extremum is governed by the system of Ritz equations $(\partial L'/\partial a_i^j)=0$. A preliminary analysis of this system shows that (a) the average particle concentrations in the upward and downward parts of the flow are equal $(a_1^\varepsilon=a_3^\varepsilon)$, in accordance with the available experimental data [7]; (b) in the fluidized bed the masses of the solid particles are conserved,

$$\int_0^1\int_0^1\int_0^1 \varepsilon w_z' r'\,dr'\,dz'\,dt'=0,$$

and then we have the familiar relation between the pressure drop in the bed and the weight of the material,

$$1-a_1^p=\frac{\rho_s gH}{p_0}a_1^\varepsilon;$$

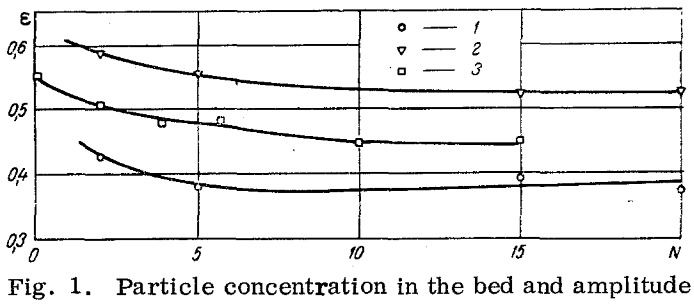

Fig. 1. Particle concentration in the bed and amplitude of the concentration fluctuations as functions of the fluidization number. 1) a_1^ε, 2) a_2^ε; 3) ε_w [9].

TABLE 1. Material Concentration and Relative Amplitude of the Fluctuations in This Concentration as Functions of the Fluidization Number

Material concentration	v_0, cm/sec	Fluidization number						
		5	15	20	30	40	50	60
a_1^{ε}	2	0,43	0,40	0,38	—	0,37	0,40	0,39
	5	0,38	0,37	0,29	0,31	—	—	0,29
$\frac{a_2^{\varepsilon}}{a_1^{\varepsilon}}$	2	1,14	1,35	1,37	—	1,43	1,00	1,03
	5	1,23	1,32	1,83	1,64	—	—	1,90

and (c) there is a linear relationship between the amplitudes of the pressure and particle-concentration fluctuations in the system,

$$a_2^p = -\frac{\rho_s g H}{p_0} a_2^{\varepsilon}.$$

These results show that this model reflects the experimental fact that the hydraulic drag (pressure drop) of a boling bed is independent of the gas filtration velocity.

The coefficients a_i^j are determined by numerically minimizing the function (9) by the method of steepest descent [8]. The solution was carried out on an MIR-1 computer.

The calculations were carried out for two materials, differing in the velocity at which fluidization begins (v_0 = 2 and 5 cm/sec), for gas filtration velocities from $2v_0$ to $60v_0$.

All the coefficients a_i^j turned out to be real, implying an oscillatory nature of the changes in all the functions over space and time. Self-oscillations of the phase velocity, pressure, and density appear in this two-phase system, although the energy is supplied to this system by a steady-state flow of a uniformly filtering gas.

The first approximation found from this solution gives a very crude description. It is interesting to compare it with experiment and thus to test the proposed formulation and solution of the problem of the hydrodynamics of a two-phase flow.

Figure 1 shows the calculated average material concentration in the bed and data on the amplitude of the concentration fluctuations. These curves were plotted for a single material (v_0 = 2 cm/sec); the experimental points shown are the average particle concentrations in the core of a bed of silica gel fluidized by air (v_0 = 2 cm/sec), obtained on the basis of measurements of the bed expansion [9].

The experimental and calculated curves are equidistant. Interestingly, the material concentration in the boiling bed changes only slightly as the velocity at which the gas filters through the bed is increased significantly. For example, when the fluidization number is increased from 2 to 20 the fluidized bed expands by no more than 20%, according to both experiment and calculation.

It is also interesting to note that the amplitude of the concentration fluctuations is slightly above its average value; this circumstance can be seen particularly clearly in Table 1, which shows the relative amplitude of the concentration fluctuations for two materials. The quantity $a_2^{\varepsilon}/a_1^{\varepsilon}$ is always larger than one. Accordingly, in a fluidized system there are regions devoid of particles in which the concentration is extremely low. These regions are gas bubbles moving upward through the bed of disperse material. According to the calculated data, these regions move at a velocity ($v_b = Nv_0 a$) which increases linearly with increasing filtration velocity. The conclusions obtained from the mathematical model agree well with available experimental data on the motion of a gas in an inhomogeneous fluidized bed and on the velocities of the gas bubbles moving through such beds.

Accordingly, this variational formulation reveals the average motion of the phases in a two-phase concentrated system of the boiling-bed type.

NOTATION

$Ar = (gd^3/\nu^2)(\rho_s - \rho_f/\rho_f)$	is the Archimedes number;
$a = v_b/Nv_0$	is the dimensionless velocity at which a gas bubble rises;
b	is the relative radius of the ascending part of the motion of the boiling bed;
d	is the particle diameter;
$E = p_0/\rho_s v_0^2$	is the Euler number;
$Fr = v_0^2/gH$	is the Froude number;
g	is the acceleration due to gravity;
H	is the bed height;
k	is the friction coefficient;
$k_1 = Hk/\rho_s v_0$	is the dimensionless friction coefficient;
n_k	is the direction cosines;
N	is the fluidization number;
p_f, p_s	are the gas and particle pressures, respectively;
$p_f' = p_f/p_0$, $p_s' = p_s/p_0$	are the corresponding dimensionless pressures;
p_0	is the atmospheric pressure;
R	is the column radius;
t	is the time;
$t' = t/T_0$	is the dimensionless time;
$T_0 = H/Nv_0$	is the time-integration inverval;
V	is the bed volume;
v_i, w_i	are the gas and particle velocities;
$v_i' = v_i/v_0$, $w_i' = w_i/v_0$	are the corresponding dimensionless velocities;
v_0	is the velocity corresponding to the beginning of fluidization;
ε	is the particle concentration;
ν	is the kinematic viscosity;
ρ_f, ρ_s	are the gas and particle densities;
$\gamma = H/R$.	

LITERATURE CITED

1. R. I. Nigmatulin, PMM, 34, No. 6 (1970).
2. J. D. Murray, J. Fluid Mech., 21, Part 3 (1965).
3. R. S. Schechter, The Variational Method in Engineering, McGraw-Hill, New York (1971).
4. P. Glansdorff and I. Prigogine, Thermodynamic Theory of Structure, Stability, and Fluctuations, Wiley-Interscience, New York (1971).
5. S. G. Mikhlin, Numerical Methods for the Variational Approach [in Russian], Nauka, Moscow (1966).
6. S. S. Zabrodskii, Hydrodynamics and Heat Transfer in a Fluidized Bed [in Russian], Gosénergoizdat (1963).
7. M. É. Aérov and O. M. Todes, Basic Hydraulics and Thermodynamics of Systems with a Steady-State Boiling Particle Bed [in Russian], Khimiya (1968).
8. A. V. Fiacco and G. P. McCormick, Nonlinear Programming, Wiley, New York (1972).
9. A. I. Tamarin, D. M. Galershnein, S. S. Zabrodskii, R. R. Khasanov, and V. P. Borisenko, Inzh.-Fiz. Zh., 23, No. 4 (1972).